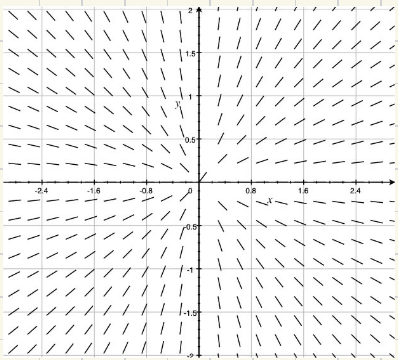
Pre-class Warm-up!!!

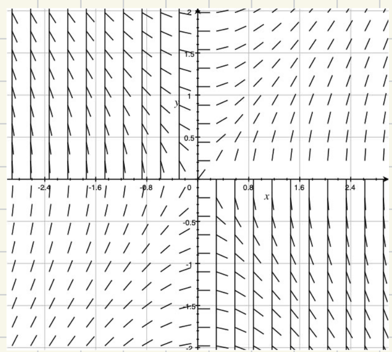
Which of the following is the slope field for the equation dy/dx = x/y ?

a.



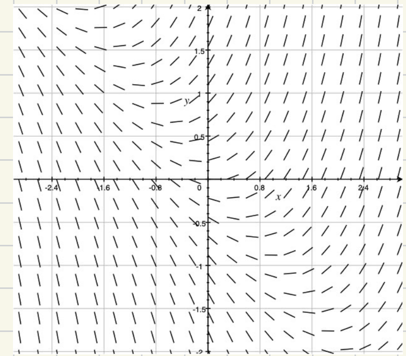
✓ b.

Ignore ||||



Ignore the vertical lines ||||

c.



Another question. Which of these is the direction field for x′ = y, y′ = x ?

Also b.

7.1 First order linear systems of equations

We learn:

- what is a linear system of equations
- how to convert a high order differential equation into a system of first order equations
- occasionally, how to convert a system of equations into a higher order differential equation
- a theorem about existence of solutions to systems of equations
- what are direction fields

Vocabulary:

- direction field, solution curve or trajectory, phase plane portrait
- homogeneous means the same thing that it did before, except in a new context.

Page 372 question 13:
Solve $x' = -2y$, $y' = 2x$, $x(0) = 1$, $y(0) = 0$.

Implicit in this is that x, y are functions of t

Solution: From eq. 1: $x'' = -2y'$

From eqn 2 $x'' = -4x$. $x'' + 4x = 0$

Char. poly: $r^2 + 4$, roots $\pm 2i$

Solution $x = A\cos 2t + B\sin 2t$

$x' = -2A\sin 2t + 2B\cos 2t = -2y$

$y = A\sin 2t - B\cos 2t$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = A\begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix} + B\begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix}$$

When $t = 0$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} + B\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

so $B = 0$, $A = 1$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 2t \\ \sin 2t \end{bmatrix}$$

A linear system of first order differential equations has the form
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}' = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

where x_i, p_{ij}, f_i are all functions of t.

Example:
$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Write this as $X' = PX + F$

Solutions are vector functions of t.

The p_{ij} need not be linear functions of t

The system is homogeneous if $F = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Page 371 question 2:
Transform the given differential equation into an equivalent system of first order differential equations.
$x'''' + 6x'' - 3x' + x = \cos 3t$

Solution: Write $x_1 = x$, $x_2 = x_1'$, $x_3 = x_2'$

$x_4 = x_3'$, $x_4' = -6x'' + 3x' - x + \cos 3t$
$= -6x_3 + 3x_2 - x_1 + \cos 3t$

This is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ -x_1 + 3x_2 - 6x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 3t \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 3t \end{bmatrix}$$

Page 371 question 10:
Same with $x'' = (1-y)x$, $y'' = (1-x)y$
(Note: this system won't be linear)

Solution $x_1 = x$ $x_2 = x_1'$
$y_1 = y$ $y_2 = y_1'$

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} x_2 \\ (1-y_1)x_1 \\ y_2 \\ (1-x_1)y_1 \end{bmatrix}$$

Section 1.6 question 46: Solve $x\,y'' + y' = 4x$

This time it's significant there is no term in y.

Put $v = \dfrac{dy}{dx} = y'$

$$y'' = \frac{dv}{dx}$$

Substitute: $x\dfrac{dv}{dx} + v = 4x$.

$$\frac{dv}{dx} + \frac{1}{x}v = 4$$

We can solve this

a. by separating the variables

✓ b. as a first order linear equation

c. by making another substitution

Then solve the equation for v that arises.

Question: Recall 1.6 question 46 above!
What does the equation $xy'' + y' = 4x$ look like when we write it as a linear system?

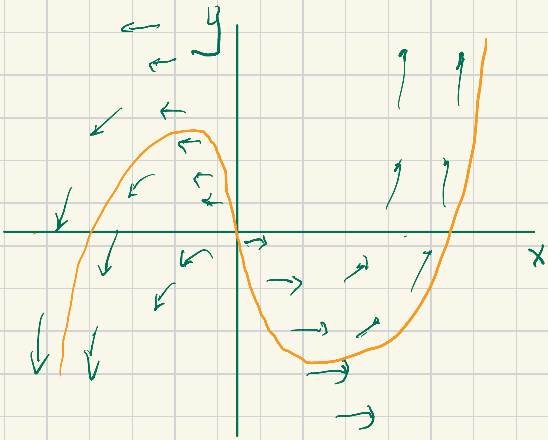
a. $\begin{bmatrix} v \\ y \end{bmatrix}' = \begin{bmatrix} v \\ -\frac{v}{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

✓ b. $\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} v \\ -\frac{v}{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

c. $\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} -\frac{v}{x} \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

d. None of the above.

Page 372 question 19:
Find the solution, and draw a direction field for
$x' = -y, \ y' = 13x + 4y; \ x(0) = 0, \ y(0) = 3.$



At each value of (x, y) draw a vector (x', y').

In this case, when $13x + 4y = 0$ then $y' = 0$

The slope of each vector is

$$\frac{y'}{x'} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dx}$$

A curve following the vectors is called a solution curve or trajectory

Solution:
$x'' = -y' = -13x - 4y = -13x + 4x'$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -e^{-2t}\sin 3t \\ e^{-2t}\,3\cos 3t + 2\sin 3t \end{bmatrix}$$

Theorem 1.

Given a first order system $X' = PX + F$ and a vector B, if the functions P and F are continuous around some number $t = a$, then there is a unique solution satisfying $X(a) = B$.